

# INFINITE DIMENSIONAL OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE AND APPLICATIONS TO HIGH ORDER HEAT-TYPE EQUATIONS

S. MAZZUCCHI

**ABSTRACT.** The definition of infinite dimensional Fresnel integrals is generalized to the case of polynomial phase functions of any degree and applied to the construction of a functional integral representation of the solution of a general class of high order heat-type equations.

*Key words:* Infinite dimensional integration, partial differential equations, representations of solutions.

*AMS classification :* 35C15, 35G05, 28C20, 47D06.

## 1. INTRODUCTION

Functional integration is a powerful tool in the study of dynamical systems [34]. One of the main examples is the Feynman-Kac formula (2), which provides a representation of the solution of the heat equation (1)

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) - V(x) u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d, V \in C_0^\infty(\mathbb{R}^d) \\ u(0, x) = u_0(x) \end{cases} \quad (1)$$

in terms of an integral with respect to the Wiener measure, the probability measure on  $C([0, t])$  associated to the Wiener process, the mathematical model of the Brownian motion [24]:

$$u(t, x) = \mathbb{E}^x \left[ e^{-\int_0^t V(\omega(s)) ds} u_0(\omega(t)) \right]. \quad (2)$$

In fact nowadays there exists an extensively developed theory connecting stochastic processes with the solution of parabolic equations associated to second-order elliptic operators [14]. On the other hand, if one considers more general PDEs, such as, for instance, the Schrödinger equation

$$\begin{cases} i \frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \Delta u(t, x) + V(x) u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d \\ u(0, x) = u_0(x) \end{cases} \quad (3)$$

which describes the time evolution of the state of a nonrelativistic quantum particle, or heat-type equations associated to higher order differential operators, as for instance:

$$\frac{\partial}{\partial t}u(t) = -\Delta^2 u(t) - V(x)u(t, x) \quad (4)$$

then the traditional theory cannot be applied. In fact it is not possible to define a stochastic Markov process that plays for Eq. (3) or Eq. (4) the same role that the Brownian motion plays for the heat equation and construct a "generalized Feynman Kac formula"

$$u(t, x) = \int_{\omega(0)=x} e^{-\int_0^t V(\omega(s))ds} u_0(\omega(t)) dP(\omega). \quad (5)$$

representing the solution of Eq. (3) or Eq. (4) in terms of a (Lebesgue integral) with respect to a probability measure  $P$  on  $\mathbb{R}^{[0,t]}$  associated to a Markov process.

Contrary to the case of the heat equation, in the case of Eq. (3) or Eq. (4) the fundamental solution  $G_t(x, y)$  for  $V = 0$ , i.e. the Green function, is not real and positive and cannot be interpreted as the density of a transition probability measure. In particular in the case  $V \equiv 0$  the Green function  $G_t(x, y)$  of the Schrödinger equation is complex, while in the case of the high order heat-type equation (4)  $G_t(x, y)$  is real but assumes both positive and negative values [19]. This fact has the troublesome consequence that the complex (resp. signed) finitely additive measure  $\mu$  on  $\Omega = \mathbb{R}^{[0,t]}$  defined on the algebra of "cylindrical sets"  $I_k \subset \Omega = \{x : [0, \infty) \rightarrow \mathbb{R}\}$  of the form

$$I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots, k\}, \quad 0 < t_1 < t_2 < \dots < t_k,$$

as

$$\mu(I_k) = \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G_{t_{j+1}-t_j}(x_{j+1}, x_j) dx_1 \dots dx_k, \quad (6)$$

cannot be extended to a  $\sigma$ -additive measure on the  $\sigma$ -algebra generated by the cylindrical sets. Indeed, if this measure exists, it would have infinite total variation.

This problem was pointed out by Cameron [13] in 1960 in the case of the Schrödinger equation and by Krylov [25] in the case of Eq. (4). These results can be regarded as particular cases of a general theorem proved by E. Thomas [35], generalizing Kolmogorov existence theorem to limits of projective systems of signed or complex measures, instead of probability ones.

In fact these negative results forbid a functional integral representation

of the solution of Eq. (3) or Eq. (4) in terms of a Lebesgue-type integral with respect to a  $\sigma$ -additive complex or signed measure with finite total variation. Consequently, the integral appearing in the generalized Feynman-Kac formula (5) has to be realized in a generalized weaker sense. One possibility is the definition of the "integral" in terms of a linear continuous functional on a suitable Banach algebra of "integrable functions", in the spirit of Rietz-Markov theorem, that states a one to one correspondence between complex bounded measures (on suitable topological spaces  $X$ ) and linear continuous functional on  $C_\infty(X)$  (the continuous functions on  $X$  vanishing at  $\infty$ ).

In the case of the Schrödinger equation this program has been extensively implemented, giving rise to several different mathematical definitions of Feynman path integrals (see [30] for a review of this topic). In particular we mention here for future reference the *Parseval approach*, introduced by Itô [22, 23] in the 60s and extensively developed in the 70s by S. Albeverio and R. Hoegh-Krohn [2, 3], and by D. Elworthy and A. Truman [15].

In the case of the parabolic equation (4) associated to the bilaplacian several different approaches have been proposed. One of the first was introduced by Krylov [25] and continued by Hochberg [19], who defined a stochastic pseudo-process whose transition probability function is not positive definite, realizing formula (5) in terms of the expectation with respect to a signed measure on  $\mathbb{R}^{[0,t]}$  with infinite total variation. For this reason the integral in (5) is not defined in Lebesgue sense, but is meant as limit of finite dimensional cylindrical approximations [7]. It is worthwhile to mention the work by D. Levin and T. Lyons [27] on rough paths, conjecturing that the signed measure (with infinite total variation) associated to the pseudo-process could exist on the quotient space of equivalence classes of paths corresponding to different parametrization of the same path.

A different approach was proposed by Funaki [16] and continued by Burdzy [10]. It is based on the construction of a complex valued stochastic process with dependent increments, obtained by composing two independent Brownian motions. In [16] formula (5), in the case where  $V = 0$ , is realized as an integral with respect to a well defined positive probability measure on a complex space, at least for a suitable class of analytic initial data  $u_0$ . These results have been further developed in [16, 20, 32] and are related to Bochner's subordination [9]. Complex valued processes, connected to PDE of the form (4) have been also proposed by other authors by means of different techniques [11, 28, 12, 33]. Recently a new construction for the solution of a general class of high

order heat-type equations has been proposed, where formula (5) has been realized as limit of expectations with respect to a sequence of particular random walks on the complex plane [8].

We also mention a completely different approach proposed by R. Léandre [26], which has some analogies with the mathematical realization of Feynman path integrals by means of white noise calculus [17].

It is worthwhile to point out that most of the results present in the literature are restricted to the case where  $V = 0$  or  $V$  is linear. The construction of a generalized Feynman-Kac type formula for the representation of the solution of high order heat type equations similar to (4) for a more general class of potentials  $V$  is still lacking.

The aim of the present paper is the construction of a Feynman-Kac formula providing a functional integral representation of the solution of a general class of high order heat-type equations of the form:

$$\frac{\partial}{\partial t}u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t, x) + V(x), \quad t \in [0, +\infty), x \in \mathbb{R}, \quad (7)$$

where  $p \in \mathbb{N}$ ,  $p > 2$ ,  $\alpha \in \mathbb{C}$  is a complex constant and  $V : \mathbb{R} \rightarrow \mathbb{C}$  a continuous bounded function.

In the spirit of the *Fresnel integral approach* for the mathematical realization of Feynman path integrals [3, 2], we introduce infinite dimensional Fresnel integrals with polynomial phase, generalizing the existing result valid for quadratic phase functions. In particular, in the case where the phase function is an homogeneous polynomial of order  $p$ , we show how this new kind of functional integral is related to the fundamental solution of Eq. (7) with  $V \equiv 0$ . This relation will be finally used in the proof of a functional integral representation of the solution of Eq. (7), for suitable class of potentials  $V$  and initial data  $u_0$ , providing a new type of generalized Feynman-Kac formula.

In section 2 we present a detailed study of the fundamental solution of Eq. (7) in the case where  $V = 0$ . In section 3 we introduce the definition of infinite dimensional Fresnel integral with polynomial phase function and show how a particular realization of this general construction can be related to the PDE (7) with  $V \equiv 0$ . In section 4 we construct a representation of the solution of (7) with  $V \neq 0$  in terms of an infinite dimensional Fresnel integral.

## 2. THE FUNDAMENTAL SOLUTION OF HIGH-ORDER HEAT-TYPE EQUATIONS

Let us consider the high order heat type equation:

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t, x) \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \in [0, +\infty) \end{cases} \quad (8)$$

where  $p \in \mathbb{N}$ ,  $p \geq 2$ , and  $\alpha \in \mathbb{C}$  is a complex constant. In the following we shall assume that  $|e^{\alpha t x^p}| \leq 1$  for all  $x \in \mathbb{R}$  and  $t \in [0, +\infty)$ . In particular, if  $p$  even we shall assume that  $\operatorname{Re}(\alpha) \leq 0$ , while if  $p$  is odd we shall take  $\alpha$  imaginary.

In the case where  $p = 2$  and  $\alpha \in \mathbb{R}$ ,  $\alpha < 0$ , we obtain the heat equation, while for  $p = 2$  and  $\alpha = i$  (8) is the Schrödinger equation. Since both cases are extensively studied, in the following we shall focus ourselves on the case where  $p \geq 3$ .

Let  $G_t^p(x, y)$  be the fundamental solution of Eq. (8), namely for  $u_0 \in S(\mathbb{R})$ , the Schwartz space of test functions, the solution of the Cauchy problem (8) is given by:

$$u(t, x) = \int_{\mathbb{R}} G_t^p(x, y) u_0(y) dy. \quad (9)$$

In particular the following equality holds:

$$G_t^p(x - y) = g_t^p(x - y),$$

where  $g_t^p \in S'(\mathbb{R})$  is the distribution defined as the Fourier transform

$$g_t^p(x) := \frac{1}{2\pi} \int e^{ikx} e^{\alpha t k^p} dk, \quad x \in \mathbb{R}. \quad (10)$$

Further The following lemma states some regularity properties of the distribution  $g_t^p$  that will be used in the next section.

**Lemma 1.** *For  $p \geq 3$  and  $t > 0$  the tempered distribution  $g_t^p \in S'(\mathbb{R})$  defined by (10) belongs to  $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ .*

*Proof.* The proof is divided into two step. We shall first prove the regularity, then we shall study the behavior of  $g_t^p(x)$  as  $x \rightarrow \infty$ .

*1. Regularity of  $g_t^p$ :* A priori  $g_t^p$  is an element of  $S'(\mathbb{R})$ , the Schwartz space of distribution, but we shall prove that  $g_t^p$  is a  $C^\infty$  function defined by an absolutely convergent Lebesgue integral.

In the case where  $p$  is even and  $\operatorname{Re}(\alpha) < 0$  the function  $k \mapsto e^{\alpha t k^p}$  is an element of  $L^1(\mathbb{R})$  and the integral (10) is absolutely convergent and defines a  $C^\infty$  function of the variable  $x \in \mathbb{R}$ .

In the case where  $\operatorname{Re}(\alpha) = 0$ , i.e.  $\alpha = ic$  with  $c \in \mathbb{R}$ , the function  $k \mapsto e^{\alpha t k^p}$  does not belong to  $L^1(\mathbb{R})$ . Let us denote by  $\psi \in S'(\mathbb{R})$

the tempered distribution defined by this map and by  $\chi_{[-R,R]}$  the characteristic function of the interval  $[-R, R] \subset \mathbb{R}$ . By the convergence of  $\chi_{[-R,R]}\psi \rightarrow \psi$  in  $S'(\mathbb{R})$  as  $R \rightarrow +\infty$  and the continuity of the Fourier transform as a map from  $S'(\mathbb{R})$  to  $S'(\mathbb{R})$  we have that  $g_t^p = \hat{\psi} = \lim_{R \rightarrow +\infty} \widehat{\chi_{[-R,R]}\psi}$ .

On the other hand, by a change in the integration path in the complex  $k$ -plane, in the case where  $p$  is even and  $c > 0$  we have:

$$\begin{aligned} g_t^p(x) &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{ikx} e^{ictk^p} dk = \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_0^R (e^{ikx} + e^{-ikx}) e^{ictk^p} dk \\ &= \lim_{R \rightarrow \infty} \frac{e^{i\pi/2p}}{2\pi} \int_0^R (e^{ie^{i\pi/2p}kx} + e^{-ie^{i\pi/2p}kx}) e^{-ctk^p} dk \\ &= \frac{e^{i\pi/2p}}{2\pi} \int_{\mathbb{R}} e^{ie^{i\pi/2p}kx} e^{-ctk^p} dk, \end{aligned} \quad (11)$$

while in the case where  $p$  is even and  $c < 0$ :

$$g_t^p(x) = \frac{e^{-i\pi/2p}}{2\pi} \int_{\mathbb{R}} e^{ie^{-i\pi/2p}kx} e^{-ctk^p} dk. \quad (12)$$

In the case where  $p$  is odd, by deforming the integration path in a different way it is possible to obtain the following representation:

$$\begin{aligned} g_t^p(x) &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{ikx} e^{ictk^p} dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}+i\eta} e^{ixz} e^{ictz^p} dz \end{aligned} \quad (13)$$

where  $\eta > 0$  if  $c > 0$  while  $\eta < 0$  if  $c < 0$ . The integrand in the second line of (13) is absolutely convergent since  $|e^{ict(Re(z)+i\eta)^p}| \sim e^{-ct\eta(Re(z))^{p-1}}$  as  $|Re(z)| \rightarrow \infty$ .

Representations (11), (12) and (13) show that  $g_t^p$  is a  $C^\infty$  function of the variable  $x$ .

*2. Behaviour of  $g_t^p$  for  $x \rightarrow \infty$*  In the second part of the proof we show that the function  $g_t^p : \mathbb{R} \rightarrow \mathbb{C}$  belongs to  $L^1(\mathbb{R})$  by studying its asymptotic behavior for  $|x| \rightarrow \infty$  by means of the stationary phase method [31, 21].

For  $x \rightarrow +\infty$ , by a change of variables we have

$$g_t^p(x) = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{x^{p/p-1}(i\xi + \alpha t \xi^p)} d\xi = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{x^{p/p-1}\phi(\xi)} d\xi, \quad (14)$$

where  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  be phase function of the exponential appearing in the integral (14), namely

$$\phi(\xi) = i\xi + \alpha t \xi^p, \quad \xi \in \mathbb{R}.$$

Let us consider first of all the cases where the phase function  $\phi$  has no stationary points on the real line, i.e. if there are no  $\xi_0 \in \mathbb{R}$  such that  $\phi'(\xi_0) = 0$ . This condition is satisfied if  $\operatorname{Re}(\alpha) \neq 0$  or if  $p$  is odd and  $\alpha = ic$ , with  $c \in \mathbb{R}$ ,  $c > 0$ . In all these cases, by an integration by parts argument, we can write:

$$\int e^{x^{\frac{p}{p-1}} \phi(\xi)} d\xi = \int \frac{1}{x^{\frac{p}{p-1}} \phi'(\xi)} \frac{d}{d\xi} e^{x^{\frac{p}{p-1}} \phi(\xi)} = \frac{1}{x^{\frac{p}{p-1}}} \int e^{x^{\frac{p}{p-1}} \phi(\xi)} \frac{\phi''(\xi)}{(\phi'(\xi))^2}.$$

By iterating this procedure, one can see that as  $x \rightarrow +\infty$  one has  $g_t^p(x) \ll (x^{\frac{p}{p-1}})^N$  for all  $N \in \mathbb{N}$ .

In the case where  $\operatorname{Re}(\alpha) = 0$ , i.e.  $\alpha = ic$  with  $c \in \mathbb{R}$ , Eq. (14) assumes the following form:

$$g_t^p(x) = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi+ct\xi^p)} d\xi, \quad x > 0.$$

If  $p$  is even, an application of the stationary phase method [31, 21] gives for  $x \rightarrow \infty$ :

$$\begin{aligned} g_t^p(x) &= \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi+ct\xi^p)} d\xi \\ &\sim e^{\operatorname{sign}(c)i\frac{\pi}{4}} \frac{x^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{-ix^{p/p-1}\frac{p-1}{p}(\frac{1}{pct})^{1/p-1}} \sqrt{\frac{(pct)^{\frac{p-2}{p-1}}}{|c|tp(p-1)}} \quad x \rightarrow \infty \end{aligned}$$

In the case where  $p$  is odd and  $c < 0$ , the same technique gives the following asymptotic behaviour for  $x \rightarrow \infty$ :

$$\begin{aligned} g_t^p(x) &= \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi+ct\xi^p)} d\xi \\ &\sim e^{-i\frac{\pi}{4}} (p-1)^{-1/2} (p|c|t)^{-\frac{1}{2(p-1)}} \frac{x^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{ix^{p/p-1}\frac{p-1}{p}(-\frac{1}{pct})^{\frac{1}{p-1}}} \end{aligned}$$

In the case where  $x \rightarrow -\infty$  and  $p$  is an even integer, then  $g_t^p$  is an even function and its asymptotic behavior for  $x \rightarrow -\infty$  coincides with the one for  $x \rightarrow +\infty$ . In the case where  $p$  is odd and  $x \rightarrow -\infty$ , we have:

$$g_t^p(x) = \frac{(-x)^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{i(-x)^{p/p-1}(-\xi+ct\xi^p)} d\xi.$$

If  $c < 0$  then the phase function  $\phi(\xi) = -\xi + ct\xi^p$  has no stationary points and for  $x \rightarrow -\infty$  we have that  $g_t^p(x) \ll x^{-N}$  for all  $N \in \mathbb{N}$ , while in the case where  $c > 0$  and  $x \rightarrow -\infty$  we have:

$$G_t(x) \sim e^{i\frac{\pi}{4}}(p-1)^{-1/2}(pct)^{-\frac{1}{2(p-1)}} \frac{(-x)^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{i(-x)^{p/p-1} \frac{1-p}{p}(pct)^{-\frac{1}{p-1}}}.$$

Eventually all this results show that if  $p > 2$  then  $g_t^p(x) \rightarrow 0$  as  $x \rightarrow \infty$ . □

**Remark 1.** *The first part of the proof of the previous lemma shows that  $g_t^p : \mathbb{R} \rightarrow \mathbb{C}$  can be extended to an entire analytic function of  $z \in \mathbb{C}$ . The analyticity of  $g_t^p$  follows by the application of Fubini's and Morera's theorems.*

**Remark 2.** *A formula similar to (11) has also been proved in [4] and applied to the study of some asymptotic properties of finite dimensional Fresnel integral with polynomial phase function.*

### 3. INFINITE DIMENSIONAL FRESNEL INTEGRALS WITH POLYNOMIAL PHASE

In [3] the definition of *infinite dimensional Fresnel integral* is introduced. The main idea is the generalization of the Parseval equality

$$\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2}\|x\|^2}}{(2\pi i)^{n/2}} f(x) dx = \int_{\mathbb{R}^n} e^{-\frac{i}{2}\|x\|^2} \hat{f}(x) dx, \quad (15)$$

(valid for Schwartz test functions  $f \in S(\mathbb{R}^n)$ ) with  $\hat{f}(x) = \int_{\mathbb{R}^n} e^{ixy} f(y) dy$  to the case where  $\mathbb{R}^n$  is replaced by a real separable infinite dimensional Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ .

Let  $\mathcal{M}(\mathcal{H})$  be the Banach space of complex Borel measure on  $\mathcal{H}$  with finite total variation, endowed with the total variation norm, denoted by  $\|\mu\|_{\mathcal{M}(\mathcal{H})}$ .  $\mathcal{M}(\mathcal{H})$  is a commutative Banach algebra under convolution, where the unit is the  $\delta$  point measure.

Let  $\mathcal{F}(\mathcal{H})$  be the space of complex functions on  $\mathcal{H}$  that are Fourier transforms of measures belonging to  $\mathcal{M}(\mathcal{H})$ , i.e. functions of the form:

$$f(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu(y), \quad x \in \mathcal{H} \quad (16)$$

for some  $\mu \in \mathcal{M}(\mathcal{H})$ . The map from  $\mathcal{M}(\mathcal{H})$  to  $\mathcal{F}(\mathcal{H})$  given by (16) is linear and one to one.

By introducing on  $\mathcal{F}(\mathcal{H})$  the norm  $\|f\|_{\mathcal{F}} = \|\mu\|_{\mathcal{M}(\mathcal{H})}$ , where  $f \in \mathcal{F}(\mathcal{H})$  is Fourier transform of  $\mu \in \mathcal{M}(\mathcal{H})$ , the map (16) becomes an isometry and  $\mathcal{F}(\mathcal{H})$  endowed with the norm  $\|\cdot\|_{\mathcal{F}}$  a commutative Banach algebra



of continuous functions.

In [3, 2] the Parseval equality (15) is generalized to the case where  $f \in \mathcal{F}(\mathcal{H})$ . The *infinite dimensional Fresnel integral* of a function  $f \in \mathcal{F}(\mathcal{H})$  is denoted by  $\tilde{\int} e^{\frac{i}{2}\|x\|^2} f(x) dx$  and defined as:

$$\tilde{\int} e^{\frac{i}{2}\|x\|^2} f(x) dx := \int_{\mathcal{H}} e^{-\frac{i}{2}\|x\|^2} d\mu(x), \quad (17)$$

where  $f(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu(y)$  and the right hand side of (17) is a well defined (absolutely convergent) Lebesgue integral.

Infinite dimensional Fresnel integrals have been successfully applied to the representation of the solution of Schrödinger equation (3) (see i.e. [3, 30] and references therein). Indeed let us consider the Hilbert space  $H_t$  of absolutely continuous maps  $\gamma : [0, t] \rightarrow \mathbb{R}^d$ , such that  $\int_0^t \dot{\gamma}(s)^2 ds < \infty$  and  $\gamma(t) = 0$ , endowed with the inner product  $\langle \gamma, \eta \rangle = \int_0^t \dot{\gamma}(s) \dot{\eta}(s) ds$ . Under the assumption that the initial datum  $u_0$  and the potential  $V$  in the Cauchy problem (3) belong to  $\mathcal{F}(\mathbb{R}^d)$ , the following functional on  $H_t$ :

$$\gamma \mapsto u_0(\gamma(0) + x) e^{-i \int_0^t V(\gamma(s) + x) ds}, \quad \gamma \in H_t, x \in \mathbb{R}^d,$$

belongs to  $\mathcal{F}(H_t)$  and the infinite dimensional Fresnel integral

$$\tilde{\int} e^{\frac{i}{2}\|\gamma\|^2} e^{-i \int_0^t V(\gamma(s) + x) ds} u_0(\gamma(0) + x) d\gamma$$

is well defined and provides a functional integral representation of the solution of the Schrödinger equation (3).

A partial generalization of the definition of infinite dimensional Fresnel integrals and of formula (17) has been developed in [5], where the quadratic phase function  $\Phi(x) = \frac{i}{2}\|x\|^2$  appearing in the exponential  $e^{\frac{i}{2}\|x\|^2}$  has been replaced with a fourth order polynomial. This new functional integral has been applied to the functional integral representation of the solution of the Schrödinger equation with a quartic potential [5, 6, 29].

In the following we are going to generalize the definition (17) to polynomial phase functions of any order and apply these *generalized Fresnel integrals* to the functional integral representation of the solution of higher order heat-type equations of the form (7).

Let us consider a real separable Banach space  $(\mathcal{B}, \|\cdot\|)$  and let  $\mathcal{M}(\mathcal{B})$  be the space of complex bounded variation measures on  $\mathcal{B}$ , endowed with the total variation norm. As remarked above,  $\mathcal{M}(\mathcal{B})$  is a Banach algebra under convolution. Let  $\mathcal{B}^*$  be the topological dual of  $\mathcal{B}$  and let

$\mathcal{F}(\mathcal{B})$  be the Banach algebra of complex-valued functions  $f : \mathcal{B}^* \rightarrow \mathbb{C}$  of the form

$$f(x) = \int_{\mathcal{B}} e^{i\langle x, y \rangle} d\mu(y), \quad x \in \mathcal{B}^*,$$

where  $\langle \cdot, \cdot \rangle$  denotes the dual pairing between  $\mathcal{B}$  and  $\mathcal{B}^*$  and  $\mu \in \mathcal{M}(\mathcal{B})$ . One has that  $\mathcal{F}(\mathcal{B})$ , endowed with the total variation norm  $\|f\|_{\mathcal{F}} := \|\mu\|_{\mathcal{M}(\mathcal{B})}$  and the pointwise multiplication is a Banach algebra of functions.

In the following we are going to define a class of linear continuous functionals on  $\mathcal{F}(\mathcal{B})$ , by generalizing the construction of infinite dimensional Fresnel integrals defined by Eq. (17).

**Definition 1.** Let  $p \in \mathbb{N}$  and let  $\Phi_p : \mathcal{B} \rightarrow \mathbb{C}$  be a continuous homogeneous map of order  $p$ , i.e. such that:

- (1)  $\Phi_p(\lambda x) = \lambda^p \Phi_p(x)$ , for all  $\lambda \in \mathbb{R}$ ,  $x \in \mathcal{B}$ ,
- (2)  $\operatorname{Re}(\Phi_p(x)) \leq 0$  for all  $x \in \mathcal{B}$ .

The infinite dimensional Fresnel integral on  $\mathcal{B}^*$  with phase function  $\Phi_p$  is the functional  $I_{\Phi_p} : \mathcal{F}(\mathcal{B}) \rightarrow \mathbb{C}$ , given by

$$I_{\Phi_p}(f) := \int_{\mathcal{B}} e^{\Phi_p(x)} d\mu(x), \quad f \in \mathcal{F}(\mathcal{B}), f(x) = \int_{\mathcal{B}} e^{i\langle x, y \rangle} d\mu(y). \quad (18)$$

By construction, one can easily verify that the functional  $I_{\Phi_p}$  is linear. Moreover, since by assumption  $\operatorname{Re}(\Phi_p(x)) \leq 0$ , the integrand in the definition of  $I_{\Phi_p}(f)$  is uniformly bounded, i.e.  $|e^{\Phi_p(x)}| \leq 1 \ \forall x \in \mathcal{B}$ , and the following inequality

$$|I_{\Phi_p}(f)| \leq \int_{\mathcal{B}} |e^{\Phi_p(x)}| d|\mu|(x) \leq \|\mu\| = \|f\|_{\mathcal{F}}$$

gives the continuity of  $I_{\Phi_p}$  in the  $\|\cdot\|_{\mathcal{F}}$ -norm. Further  $I_{\Phi_p}(1) = 1$ . We summarize these properties in the following proposition.

**Proposition 1.** The space  $\mathcal{F}(\mathcal{B})$  of Fresnel integrable functions is a Banach function algebra in the norm  $\|\cdot\|_{\mathcal{F}}$ . The infinite dimensional Fresnel integral with phase function  $\Phi_p$  is a continuous bounded linear functional  $I_{\Phi_p}$  on  $\mathcal{F}(\mathcal{B})$  such that  $|I_{\Phi_p}(f)| \leq \|f\|_{\mathcal{F}}$  and normalized so that  $I_{\Phi_p}(1) = 1$ .

We can now give an interesting example of infinite dimensional Fresnel integral with polynomial phase function.

Fixed a  $p \in \mathbb{N}$ , with  $p \geq 2$ , let us consider the Banach space  $\mathcal{B}_p$  of absolutely continuous maps  $\gamma : [0, t] \rightarrow \mathbb{R}$ , with  $\gamma(t) = 0$  and a weak

derivative  $\dot{\gamma}$  belonging to  $L^p([0, t])$ , endowed with the norm:

$$\|\gamma\|_{\mathcal{B}_p} = \left( \int_0^t |\dot{\gamma}(s)|^p ds \right)^{1/p}.$$

One has that the transformation  $T : \mathcal{B}_p \rightarrow L^p([0, t])$  mapping an element  $\gamma \in \mathcal{B}_p$  to its weak derivative  $\dot{\gamma}$  is an isomorphism. Its inverse  $T^{-1} : L^p([0, t]) \rightarrow \mathcal{B}_p$  is given by:

$$T^{-1}(v)(s) = - \int_s^t v(u) du \quad v \in L^p([0, t]). \quad (19)$$

Analogously the dual space  $\mathcal{B}_p^*$  is isomorphic to  $L_q([0, t]) = (L_p([0, t]))^*$ , where the pairing between an element  $\eta \in \mathcal{B}_p^*$  and  $\gamma \in \mathcal{B}_p$  is given by:

$$\langle \eta, \gamma \rangle = \int_0^t \dot{\eta}(s) \dot{\gamma}(s) ds \quad \dot{\eta} \in L_q([0, t]), \gamma \in \mathcal{B}_p.$$

By means of the map (19) it is simple to verify that  $\mathcal{B}_p^*$  is isomorphic to  $\mathcal{B}_q$ .

Let  $\mathcal{F}(\mathcal{B}_q)$  be the space of functions  $f : \mathcal{B}_q \rightarrow \mathbb{C}$  of the form

$$f(\eta) = \int_{\mathcal{B}_p} e^{i \int_0^t \dot{\eta}(s) \dot{\gamma}(s) ds} d\mu_f(\gamma), \quad \eta \in \mathcal{B}_q, \mu_f \in \mathcal{M}(\mathcal{B}_p).$$

Let  $\Phi_p : \mathcal{B}_p \rightarrow \mathbb{C}$  be the phase function

$$\Phi_p(\gamma) := (-1)^p \alpha \int_0^t \dot{\gamma}(s)^p ds,$$

where  $\alpha \in \mathbb{C}$  is a complex constant such that

- $Re(\alpha) \leq 0$  if  $p$  is even,
- $Re(\alpha) = 0$  if  $p$  is odd.

In particular, in the case where  $p$  is even one has that  $\Phi_p(\gamma)$  is proportional to the  $\mathcal{B}_p$ -norm of the vector  $\gamma$ .

Under these assumptions the infinite dimensional Fresnel integral with phase function  $\Phi_p$  of a function  $f \in \mathcal{F}(\mathcal{B}_q)$ ,  $f = \hat{\mu}_f$ , is given by:

$$I_{\Phi_p}(f) = \int_{\mathcal{B}_p} e^{(-1)^p \alpha \int_0^t \dot{\gamma}(s)^p ds} d\mu_f(\gamma). \quad (20)$$

The following lemma creates an interesting connection between the functional  $I_{\Phi_p}$  and the high order pde (8).

**Lemma 2.** *Let  $f : \mathcal{B}_q \rightarrow \mathbb{C}$  be a cylindric function of the following form:*

$$f(\eta) = F(\eta(t_1), \eta(t_2), \dots, \eta(t_n)), \quad \eta \in \mathcal{B}_q,$$

with  $0 \leq t_1 < t_2 < \dots < t_n < t$  and  $F : \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $F \in \mathcal{F}(\mathbb{R}^n)$ :

$$F(x_1, x_2, \dots, x_n) = \int_{\mathbb{R}^n} e^{i \sum_{k=1}^n y_k x_k} d\nu_F(y_1, \dots, y_n), \quad \nu_F \in \mathcal{M}(\mathbb{R}^n).$$

Then  $f \in \mathcal{F}(\mathcal{B}_p)$  and its infinite dimensional Fresnel integral with phase function  $\Phi_p$  is given by

$$I_{\Phi_p}(f) = \int_{\mathbb{R}^n} F(x_1, x_2, \dots, x_n) \Pi_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}, x_k) dx_k, \quad (21)$$

where  $x_{n+1} \equiv 0$ ,  $t_{n+1} \equiv t$  and  $G_s^p$  is the fundamental solution (9) of the high order heat-type equation (8).

**Remark 3.** One can easily see that lemma 1, stating the summability of the function  $g_t^p$  defined by (10), assures that the integral (21) is absolutely convergent, indeed:

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x_1, x_2, \dots, x_n)| \Pi_{k=1}^n |G_{t_{k+1}-t_k}^p(x_{k+1}, x_k)| dx_k \\ \leq \|F\|_{\mathcal{F}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Pi_{k=1}^n |g_{t_{k+1}-t_k}^p(y_k)| dy_k < \infty \end{aligned} \quad (22)$$

where  $y_k = x_{k+1} - x_k$  and we have used the inequality  $\|F\|_\infty \leq \|F\|_{\mathcal{F}(\mathbb{R}^n)}$ .

*Proof.* We have

$$f(\eta) = F(\eta(t_1), \eta(t_2), \dots, \eta(t_n)) = \int_{\mathbb{R}^n} e^{i \sum_{k=1}^n y_k \eta(t_k)} d\nu_F(y_1, \dots, y_n), \quad \eta \in \mathcal{B}_q.$$

One can easily verify that  $f \in \mathcal{F}(\mathcal{B}_q)$ , indeed:

$$e^{iy\eta(s)} = \int_{\mathcal{B}_p} e^{i\langle \eta, \gamma \rangle} \delta_{y v_s}(\gamma),$$

where  $v_s \in \mathcal{B}_p$  is the element of  $\mathcal{B}_p$  defined by

$$\langle \eta, v_s \rangle = \eta(s), \quad \forall \eta \in \mathcal{B}_q,$$

which have the following form

$$v_s(\tau) = \chi_{[0,s]}(t-s) + \chi_{(s,t]}(t-\tau)s.$$

We then have:

$$\begin{aligned}
I_{\Phi_p}(f) &= \int_{\mathbb{R}^n} e^{(-1)^p \alpha \int_0^t (\sum_{k=1}^n y_k \dot{v}_{t_k}(\tau))^p d\tau} d\nu_F(y_1, \dots, y_n) \\
&= \int_{\mathbb{R}^n} e^{\alpha \int_0^t (\sum_{k=1}^n y_k \chi_{(t_k, t]}(\tau))^p d\tau} d\nu_F(y_1, \dots, y_n) \\
&= \int_{\mathbb{R}^n} e^{\alpha \int_0^t (\sum_{k=1}^n \chi_{(t_k, t_{k+1}]}(\tau) \sum_{j=1}^k y_j)^p d\tau} d\nu_F(y_1, \dots, y_n) \\
&= \int_{\mathbb{R}^n} e^{\alpha \sum_{k=1}^n (\sum_{j=1}^k y_j)^p (t_{k+1} - t_k)} d\nu_F(y_1, \dots, y_n) \tag{23}
\end{aligned}$$

On the other hand the last line of Eq. (23) coincides with

$$\int_{\mathbb{R}^n} F(x_1, x_2, \dots, x_n) \Pi_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}, x_k) dx_k,$$

indeed, by a change of variables and Fubini's theorem, the latter is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-i \sum_{k=1}^n \xi_k \sum_{l=1}^k y_l} \Pi_{k=1}^n g_{t_{k+1}-t_k}^p(\xi_k) d\xi_k \right) d\nu_F(y_1, \dots, y_n) \\
&= \int_{\mathbb{R}^n} e^{\alpha \sum_{k=1}^n (\sum_{l=1}^k y_l)^p (t_{k+1} - t_k)} d\nu_F(y_1, \dots, y_n).
\end{aligned}$$

□

The previous lemma gives the following result:

**Corollary 1.** *Let  $u_0 \in \mathcal{F}(\mathbb{R})$ . Then the cylindric function  $f_0 : \mathcal{B}_q \rightarrow \mathbb{C}$  defined by*

$$f_0(\eta) := u_0(x + \eta(0)), \quad x \in \mathbb{R}, \eta \in \mathcal{B}_q,$$

*belongs to  $\mathcal{F}(\mathcal{B}_q)$  and its infinite dimensional Fresnel integral with phase function  $\Phi_p$  provides a representation for the solution of the Cauchy problem (8), in the sense that the function  $u(t, x) := I_{\Phi_p}(f_0)$  has the form  $u(t, x) = \int_{\mathbb{R}} G_t(x, y) u_0(y) dy$ .*

#### 4. A GENERALIZED FEYNMAN-KAC FORMULA

In the present section, we consider a Cauchy problem of the form

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t, x) + V(x) u(t, x) \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \in [0, +\infty) \end{cases} \tag{24}$$

where  $p \in \mathbb{N}$ ,  $p \geq 2$ , and  $\alpha \in \mathbb{C}$  is a complex constant such that  $|e^{\alpha t x^p}| \leq 1$  for all  $x \in \mathbb{R}, t \in [0, +\infty)$ , while  $V : \mathbb{R} \rightarrow \mathbb{C}$  is a bounded continuous function. Under these assumption the Cauchy problem (24)

is well posed on  $L^2(\mathbb{R})$ . Indeed the operator  $\mathcal{D}_p : D(\mathcal{D}_p) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  defined by

$$\begin{aligned} D(\mathcal{D}_p) &:= H^p = \{u \in L^2(\mathbb{R}), k \mapsto k^p \hat{u}(k) \in L^2\}, \\ \widehat{\mathcal{D}_p u}(k) &:= k^p \hat{u}(k), u \in D(\mathcal{D}_p), \end{aligned}$$

( $\hat{u}$  denoting the Fourier transform of  $u$ ) is self-adjoint. For  $\alpha \in \mathbb{C}$ , with  $|e^{\alpha t x^p}| \leq 1$  for all  $x \in \mathbb{R}, t \in [0, +\infty)$ , one has that the operator  $A := \alpha \mathcal{D}_p$  generates a strongly continuous semigroup  $(e^{tA})_{t \geq 0}$  on  $L^2(\mathbb{R})$ . By denoting with  $B : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  the bounded multiplication operator defined by

$$Bu(x) = V(x)u(x), \quad u \in L^2(\mathbb{R}),$$

one has that the operator sum  $A + B : D(A) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on  $L^2(\mathbb{R})$ . Moreover, given a  $u \in L^2(\mathbb{R})$ , the vector  $T(t)u$  can be computed by means of the convergent (in the  $L^2(\mathbb{R})$ -norm) Dyson series (see [18], Th. 13.4.1):

$$T(t)u = \sum_{n=0}^{\infty} S_n(t)u, \quad (25)$$

where  $S_0(t)u = e^{tA}u$  and  $S_n(t)u = \int_0^t e^{(t-s)A} V S_{n-1}(s)u ds$ . By passing to a subsequence, the series above converges also a.e. in  $x \in \mathbb{R}$  giving

$$\begin{aligned} T(t)u(x) &= \sum_{n=0}^{\infty} \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \int_{\mathbb{R}^{n+1}} V(x_1) \dots V(x_n) G_{t-s_n}(x, x_n) \\ &G_{s_n-s_{n-1}}(x_n, x_{n-1}) \dots G_{s_1}(x_1, x_0) u_0(x_0) dx_0 \dots dx_n ds_1 \dots ds_n, \quad \text{a.e. } x \in \mathbb{R}. \end{aligned} \quad (26)$$

Under suitable assumptions on the initial datum  $u_0$  and the potential  $V$ , we are going to construct a representation of the solution of equation (24) in  $L^2(\mathbb{R})$  in terms of an infinite dimensional oscillatory integral with polynomial phase.

**Theorem 1.** *Let  $u_0 \in \mathcal{F}(\mathbb{R}) \cap L^2(\mathbb{R})$  and  $V \in \mathcal{F}(\mathbb{R})$ , with  $u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu_0(y)$  and  $V(x) = \int_{\mathbb{R}} e^{ixy} d\nu(y)$ ,  $\mu_0, \nu \in \mathcal{M}(\mathbb{R})$ . Then the functional  $f_{t,x} : \mathcal{B}_q \rightarrow \mathbb{C}$  defined by*

$$f_{t,x}(\eta) := u_0(x + \eta(0)) e^{\int_0^t V(x + \eta(s)) ds}, \quad x \in \mathbb{R}, \eta \in \mathcal{B}_q, \quad (27)$$

*belongs to  $\mathcal{F}(\mathcal{B}_q)$  and its infinite dimensional Fresnel integral with phase function  $\Phi_p$  provides a representation for the solution of the Cauchy problem (24).*

*Proof.* Let  $\mu_V \mathcal{M}(\mathcal{B}_p)$  be the measure defined by

$$\int_{B_p} f(\gamma) d\mu_V(\gamma) = \int_0^t \int_{\mathbb{R}} e^{ixy} f(y v_s) d\nu(y) ds, \quad f \in C_b(\mathcal{B}_p),$$

where  $v_s \in \mathcal{B}_p$  is the function  $v_s(\tau) = \chi_{[0,s]}(\tau)(t-s) + \chi_{(s,t]}(\tau-s)s$ . One can easily verify that  $\|\mu_V\|_{\mathcal{M}(\mathcal{B}_p)} \leq t\|\nu\|_{\mathcal{M}(\mathbb{R})}$  and the map  $\eta \in \mathcal{B}_q \mapsto \int_0^t V(x + \eta(s)) ds$  is the Fourier transform of  $\mu_V$ . Analogously the map  $\eta \in \mathcal{B}_q \mapsto \exp(\int_0^t V(x + \eta(s)) ds)$  is the Fourier transform of the measure  $\nu_V \in \mathcal{M}(\mathcal{B}_p)$  given by  $\nu_V = \sum_{n=0}^{\infty} \frac{1}{n!} (\mu_V)^n$ , where  $(\mu_V)^n$  denotes the convolution of  $\mu_V$  with itself  $n$ -times, i.e. for any  $f \in C_b(\mathcal{B}_p)$ :

$$\int_{B_p} f(\gamma) d\nu_V(\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B_p \times B_p \times \dots \times B_p} f(\gamma_1 + \dots + \gamma_n) d\mu_V(\gamma_1) \dots d\mu_V(\gamma_n).$$

The series is convergent in the  $\mathcal{M}(\mathcal{B}_p)$ -norm and one has  $\|\nu_V\|_{\mathcal{M}(\mathcal{B}_p)} \leq e^{t\|\nu\|_{\mathcal{M}(\mathbb{R})}}$ . Further, by lemma 2 the cylindric function  $\eta \in \mathcal{B}_q \mapsto u_0(x + \eta(0))$  is an element of  $\mathcal{F}(\mathcal{B}_q)$ , namely the Fourier transform of the measure  $\mu_{u_0}$  defined by

$$\int_{B_p} f(\gamma) d\mu_{u_0}(\gamma) = \int_{\mathbb{R}} e^{ixy} f(y v_0) d\mu_0(y), \quad f \in C_b(\mathcal{B}_p).$$

We can then conclude that the map  $f_{t,x} : \mathcal{B}_q \rightarrow \mathbb{C}$  defined by (27) belongs to  $\mathcal{F}(\mathcal{B}_q)$  and its infinite dimensional Fresnel integral  $I_{\Phi_p}(f_{t,x})$  with phase function  $\Phi_p$  is given by

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B_p} e^{(-1)^p \alpha \int_0^y \dot{\gamma}(s)^p ds} d\mu_{u_0} * \mu_V * \dots * \mu_V \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t \dots \int_0^t I_{\Phi_p}(u_0(x + \eta(0))V(x + \eta(s_1)) \dots V(x + \eta(s_n))) ds_1 \dots ds_n \end{aligned}$$

By the symmetry of the integrand the latter is equal to

$$\sum_{n=0}^{\infty} \int \dots \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} I_{\Phi_p}(u_0(x + \eta(0))V(x + \eta(s_1)) \dots V(x + \eta(s_n))) ds_1 \dots ds_n$$

By lemma 2 we finally obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \int \dots \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \int_{\mathbb{R}^{n+1}} u_0(x + x_0) V(x + x_1) \dots V(x + x_n) G_{s_1}(x_1, x_0) \\ & \quad G_{s_2-s_1}(x_2, x_1) \dots G_{t-s_n}(0, x_n) dx_0 dx_1 \dots dx_n ds_1 \dots ds_n, \end{aligned}$$

that, as one can easily verify by means of a change of variables argument, coincides with the Dyson series (26) for the solution of the high-order PDE (24).  $\square$

## ACKNOWLEDGMENTS

Many interesting discussions with Prof. S. Albeverio, S. Bonaccorsi, G. Da Prato and L. Tubaro are gratefully acknowledged, as well as the financial support of CIRM-Fondazione Bruno Kessler to the project *Functional integration and applications to quantum dynamical systems*.

## REFERENCES

- [1] S. Albeverio and Z. Brzeźniak. Finite-dimensional approximation approach to oscillatory integrals and stationary phase in infinite dimensions. *J. Funct. Anal.*, 113(1): 177-244, 1993.
- [2] S. Albeverio and R. Høegh-Krohn. Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics. *Invent. Math.*, 40(1):59-106, 1977.
- [3] S. Albeverio, R. Høegh-Krohn, S. Mazzucchi, Mathematical theory of Feynman path integrals - An Introduction. 2nd corrected and enlarged edition. Lecture Notes in Mathematics, Vol. 523. Springer, Berlin, (2008).
- [4] S. Albeverio, S. Mazzucchi, Generalized Fresnel integrals. *Bull. Sci. Math.* 129 (2005), no. 1, 123.
- [5] S. Albeverio, S. Mazzucchi, Feynman path integrals for polynomially growing potentials, *J. Funct. Anal.* 221 no.1, 83–121 (2005).
- [6] S. Albeverio, S. Mazzucchi, The time-dependent quartic oscillator Feynman path integral approach. *J. Funct. Anal.* 238 (2006), no. 2, 471-488.
- [7] S. Beghin, K. Hochberg, E. Orsingher. Conditional maximal distributions of processes related to higher-order heat-type equations. *Stochastic Process. Appl.* 85 (2000), no. 2, 209–223.
- [8] S. Bonaccorsi, S. Mazzucchi. High order heat-type equations and random walks on the complex plane. arXiv:1402.6140 [math.PR] (2014).
- [9] S. Bochner. Harmonic analysis and the theory of probability. University of California Press, Berkeley and Los Angeles, (1955).
- [10] K. Burdzy. Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 67–87. Birkhäuser Boston, Boston, MA, 1993.
- [11] K. Burdzy and A. Małdrecki. An asymptotically 4-stable process. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, number Special Issue, pages 97–117, 1995.
- [12] K. Burdzy, A. Małdrecki. Ito formula for an asymptotically 4-stable process. *Ann. Appl. Probab.* 6 (1996), no. 1, 200–217.
- [13] R.H. Cameron. A family of integrals serving to connect the Wiener and Feynman integrals, *J. Math. and Phys.* 39, 126–140 (1960).
- [14] E. B. Dynkin. Theory of Markov processes. Dover Publications, Inc., Mineola, NY, 2006.



- [15] D. Elworthy and A. Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. *Ann. Inst. H. Poincaré Phys. Théor.*, 41(2):115–142, 1984.
- [16] T. Funaki. Probabilistic construction of the solution of some higher order parabolic differential equation. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(5):176–179, 1979.
- [17] T. Hida, H.H. Kuo, J. Potthoff, L. Streit. *White Noise*. Kluwer, Dordrecht (1995).
- [18] E. Hille, R. S: Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.
- [19] K. J. Hochberg. A signed measure on path space related to Wiener measure. *Ann. Probab.*, 6(3):433–458, 1978.
- [20] K. Hochberg, E. Orsingher. Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. *J. Theoret. Probab.* 9 (1996), no. 2, 511–532.
- [21] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*. Reprint of the second (1990) edition. *Classics in Mathematics*. Springer-Verlag, Berlin, 2003.
- [22] K. Itô. Wiener integral and Feynman integral. *Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability*. Vol 2, pp. 227–238, California Univ. Press, Berkeley, 1961.
- [23] K. Itô. Generalized uniform complex measures in the hilbertian metric space with their applications to the Feynman path integral. *Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability*. Vol 2, part 1, pp. 145–161, California Univ. Press, Berkeley, 1967.
- [24] I. Karatzas, S.E. Shreve. *Brownian motion and stochastic calculus*. Springer-Verlag, New York, 1991
- [25] V. J. Krylov. Some properties of the distribution corresponding to the equation  $\partial u / \partial t = (-1)^{q+1} \partial^{2q} u / \partial x^{2q}$ . *Soviet Math. Dokl.*, 1:760–763, 1960.
- [26] R. Léandre. Stochastic analysis without probability: study of some basic tools. *J. Pseudo-Differ. Oper. Appl.* 1 (2010), no. 4, 389–400.
- [27] D. Levin, T. Lyons. A signed measure on rough paths associated to a PDE of high order: results and conjectures. *Rev. Mat. Iberoam.* 25 (2009), no. 3, 971–994.
- [28] A. Madrecki, M. Rybaczuk. New Feynman-Kac type formula. *Rep. Math. Phys.* 32 (1993), no. 3, 301–327.
- [29] S. Mazzucchi, Feynman path integrals for the inverse quartic oscillator. *J. Math. Phys.* 49 (2008), no. 9, 093502, 15 pp.
- [30] S. Mazzucchi. *Mathematical Feynman Path Integrals and Applications*. World Scientific Publishing, Singapore (2009)
- [31] J. D. Murray, *Asymptotic analysis*. Clarendon Press, Oxford, 1974.
- [32] E. Orsingher, X. Zhao. Iterated processes and their applications to higher order differential equations. *Acta Math. Sin. (Engl. Ser.)* 15 (1999), no. 2, 173–180.
- [33] P. Sainty. Construction of a complex-valued fractional Brownian motion of order  $N$ . *J. Math. Phys.*, 33(9):3128–3149, 1992.
- [34] B. Simon, *Functional integration and quantum physics*. Second edition. AMS Chelsea Publishing, Providence, RI, 2005.

- [35] E. Thomas, Projective limits of complex measures and martingale convergence. Probab. Theory Related Fields 119 (2001), no. 4, 579588.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, 38123 POVO, ITALIA